An affine Weyl group approach to the eight-parameter discrete Painlevé equation

2001 J. Phys. A: Math. Gen. 34 10523

(http://iopscience.iop.org/0305-4470/34/48/316)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 10523–10532 PII: S0305-4470(01)23712-3

An affine Weyl group approach to the eight-parameter discrete Painlevé equation

Y Ohta[1], **A Ramani**[2] **and B Grammaticos**[3]

[1] Department of Applied Mathematics, Faculty of Engineering, Hiroshima University, 1-4-1 Kagamiyama, Higashi-Hiroshima 739-8527, Japan
[2] CPT, Ecole Polytechnique, CNRS, UMR 7644, 91128 Palaiseau, France
[3] GMPIB, Université Paris VII, Tour 24-14, 5[e] étage, case 7021, 75251 Paris, France

Received 26 January 2001
Published 23 November 2001
Online at stacks.iop.org/JPhysA/34/10523

Abstract

We present a geometrical construction of the eight-parameter discrete Painlevé equations. Our starting point is the $E_8^{(1)}$ affine Weyl group. We assume that the multi-dimensional τ-function lives on the vertices of the weight lattice of this group. We derive the bilinear equations related to the discrete Painlevé equation in the form of nonautonomous Hirota–Miwa equations and the elementary Miura transformations. The compatibility condition of the various Miura's that can be written leads to three types of equations: difference, multiplicative (q) and another type where the parameters and the independent variable enter through the arguments of elliptic functions. We write explicitly the discrete equations in the first two cases and produce their degeneration through coalescence of parameters.

PACS numbers: 02.30.Ik, 02.30.-f

1. Introduction

Discrete Painlevé (d-$\mathbb{P}$) equations are far more complex (and more fundamental) than their continuous counterparts. Soon after their discovery [1] it became clear that d-$\mathbb{P}$ exist in two flavours, difference (δ-) equations and multiplicative (q-) equations, and that there are many more than the six canonical continuous Painlevé (c-$\mathbb{P}$) equations [2]. The latter fact led to a nomenclature problem: since the integrable, nonautonomous mappings which are the d-$\mathbb{P}$ were named after their continuous limits, which are c-$\mathbb{P}$, we were faced with a proliferation of discrete versions of $\mathbb{P}$, in particular for the low-parameter ones. This was taken care of partially by (a) finding correspondences between equations and (b) by showing that some of the low-parameter d-$\mathbb{P}$ were indeed reductions of richer systems. However, the problem was far from being solved and thus the question of classification became urgent.

The key to the classification of discrete Painlevé equations was to be found in a geometrical approach [3]. This was suggested by the observation that (almost but not quite all) the d-$\mathbb{P}$ have the property of self-duality: the same equation governs the evolution along the independent variable and along the Schlesinger-induced changes of parameters [4]. Moreover, the observation that some of the difference $\mathbb{P}$ are just contiguity relations of continuous $\mathbb{P}$ suggested that the geometrical description had to be given in terms of affine Weyl groups, just as in the continuous case. This was first proposed in [5] under the name of the 'Grand Scheme' description of d-$\mathbb{P}$. The whole degeneration pattern linked to affine Weyl groups, starting from the exceptional group E_8, was empirically associated with the various discrete $\mathbb{P}$ [6]. Recently, it has been put on a rigorous basis thanks to the work of Sakai [7]. He was in fact the first to show explicitly that a third type of discrete $\mathbb{P}$ did exist, one where the parameters and the independent variable enter through the arguments of elliptic functions (a fact that we had anticipated on an intuitive, nonrigorous, basis).

Once the geometrical framework is fixed our task is far from finished. In order to derive the d-$\mathbb{P}$ it does not suffice to say that their τ-functions live on the (weight) lattice of some affine Weyl group. One must derive the bilinear equations which govern the evolutions. These bilinear systems turn out to be nonautonomous Hirota–Miwa [8] equations (the compatibility of which must be assessed). Next one must introduce the elementary Miura transformations and, *choosing an adequate path*, obtain the nonlinear d-$\mathbb{P}$. The proliferation of the d-$\mathbb{P}$ is thus related not only to the abundance of the possible geometries but also to the fact that within each of them one can define more than one evolution leading to a second-order system.

Since historically almost all the d-$\mathbb{P}$ were obtained before their geometrical classification, the approach based on affine Weyl groups has not been used in order to derive the d-$\mathbb{P}$. As a matter of fact the discrete forms of the d-$\mathbb{P}$ up to q-P_V were derived through a direct method (deautonomization of a QRT form using the singularity confinement [9] criterion, a procedure later confirmed with the aid of low-growth property [10]). They were shown later to be described by various affine Weyl groups up to and including $E_6^{(1)}$. Much later, the forms of q-P_{VI} and δ-P_V [11] were obtained as an offshoot of the study of the quadratic relations of c- and d-$\mathbb{P}$ [12]. These two equations were recently shown to be described by the $E_7^{(1)}$ [13] affine Weyl group. Clearly what was missing was the explicit form of the system related to E_8. The complexity of these equations precludes any direct, brute-force, treatment and, in fact, the geometrical description seems the only available approach. In what follows we shall show how, based on the geometry of the affine Weyl group $E_8^{(1)}$, one can derive the explicit forms of q-P_{VI} and δ-P_V. We show that the richness of this exceptional group makes possible the existence of an 'elliptic' discrete $\mathbb{P}$. However, for the latter one can only present the bilinear form and the Miura transformation, the full nonlinear expression corresponding to prohibitively long calculations.

2. The geometry of the $E_8^{(1)}$ weight lattice

Our various studies in the framework of what we have dubbed the Grand Scheme have shown that the space pertinent to the description of a discrete $\mathbb{P}$ equation and its various Schlesingers is the weight lattice of an affine Weyl group, i.e. the dual of the root system. In this paper we shall consider the geometry of the space associated with E_8. Our basic assumption is that the τ-functions live on the points of the weight lattice of $E_8^{(1)}$. The coordinates of these points, in the basis we consider, are either *all* integers or *all* half-integers, with the additional constraint that the sum of all coordinates is even. The origin obviously satisfies these requirements. By considering its nearest neighbours (NNs) we can thus find the smallest vectors that span the lattice. It turns out that the origin has 240 NN τ that define 120 directions along which vectors relating NN τ exist. We must point out here that the adjective *nearest* does not really apply to these vectors which are actually the *smallest* ones; still we will call them NVs for 'nearestneighbour-connecting vectors', a shorthand the reason for which will soon become obvious. The 240 NNs of the origin have the following form. Some of them have two coordinates $a_i = \pm 1$, $a_j = \pm 1$ while the other six vanish: clearly there are 112 of these, four for each choice of $i \neq j \in \{1, \ldots, 8\}$, (defining 56 directions where NVs exist). Note that their squared distance from the origin is 2, and thus the squared length of a NV is 2. The others have all the coordinates nonzero and of absolute value $1/2$, but with either sign. Again the squared distance of each of these points from the origin is $8(1/4) = 2$. There are only 128 such NNs, and not 256 because of the selection rule that the sum of the coordinates must be even, which means that the number of negative coordinates must be even. This defines 64 more directions where NVs exist. Though the 120 NVs, in this specific basis, seem to belong to two classes, this is not true; it is a pure artifact of the basis. In fact the NVs correspond to each other by the symmetries of the underlying finite group E_8. One way to convince oneself of this is to notice that, not only do they all have the same squared length 2, but if we compute the scalar product of a NV of either class with all the 119 others, we find that 63 are orthogonal, while the 56 others have a scalar product ± 1. Note that we never bother to assign a specific sign to an NV: only its direction and length are of interest, so there are indeed 120 of them. In fact, there is no consistent way to orient them so that the scalar product of two nonorthogonal NVs is always 1, or always -1. Of course the whole argument presented here is not specific to the origin: every τ has 240 NNs, along the 120 directions defined by the NVs.

Having defined the NNs and NVs we turn to the next-nearest neighbours (NNNs) of a given τ. We can reach them by moving away from this τ by a vector which is as small as possible a sum of NVs. This turns out to be the case if we add two orthogonal NVs (since the sum of two NVs with scalar product -1 is again an NV). So the length of such a NNV is 2, since its squared length is 4. It turns out that there are 1080 such vectors (up to an arbitrary sign) and 2160 NNNs of a given τ. This number is obtained by considering the $120 \times 63/2$ pairs of mutually orthogonal NVs, with either *relative* sign, and ignoring the *global* sign for NNVs, so we multiply by 2 for them, and by 4 to find all NNNs. Each NNV, however, is obtained from seven distinct such pairs, as can be shown in a straightforward way. For instance the NNV $(2, 0, 0, 0, 0, 0, 0, 0)$ is obtained from the seven pairs of NVs $\{(1, 0, \ldots, 1, \ldots, 0), (1, 0, \ldots, -1, \ldots, 0)\}$ where the ± 1 are at any of the seven last positions. Again let us stress that though this NNV looks unique, this is due to the particular basis we chose. All NNVs are fully equivalent, corresponding to each other through the symmetries of the finite group E_8. In this basis they seem to come in three classes, eight similar to the one mentioned above, 560 with 4 zero coordinates and 4 coordinates ± 1 (defining 70 choices for the positions of the nonzero coordinates and a factor 8 for three relative signs since we ignore the global sign) and finally 512 with one coordinate $\pm 3/2$ (say $-3/2$) in either of the eight positions, and seven coordinates $\pm 1/2$ with only six free signs since the sum must be even (so there must be an odd number of plus signs).

3. Nonlinear variables, Hirota–Miwa equations and contiguity relations

In order to introduce the nonlinear variables (for which we will use the symbols X or Y) we will make the assumption that they are defined at points of the lattice which are midpoints between one τ and one of its NNNs. For example, between the origin and its NNN $(2, 0, 0, 0, 0, 0, 0, 0)$ we have a nonlinear variable X defined at the point $(1, 0, 0, 0, 0, 0, 0, 0)$. It can be easily shown that X (and in fact any other such point) is at the midpoint not only of the original pair, but of exactly eight pairs of τ sites which are in NNN position with respect to each other (but not, in general, NNN of the origin). The eight pairs in this precise example are the original one $\{(0, 0, 0, 0, 0, 0, 0, 0), (2, 0, 0, 0, 0, 0, 0, 0)\}$ and seven of the form $\{(1, 0, \ldots, 1, \ldots, 0), (1, 0, \ldots, -1, \ldots, 0)\}$, etc, where the second nonvanishing coordinates is at any of the seven last positions. The eight vectors joining the two sites of each pair are all distinct NNVs (their length is indeed 2). One can easily see that any two of them are orthogonal. Thus there is no consistent orientation choice for these vectors.

The next step is to relate the nonlinear variable X to the τ. For each X we have eight NNVs and we can introduce eight quantities C_i which are the scalar products of these vectors and the position vector $\overrightarrow{O'X}$. (Note here that the origin O' of this position vector need not coincide with the origin of coordinates: it may well be shifted by eight arbitrary numbers α_i). However, as we explained above, the orientations are not determined, consequently there exists an arbitrariness in the definition of the sign of each C_i: we can change any of the C_i to its opposite value. Next, we introduce the quantities ϕ_i which are the products of the two τ at the ends of each vector, and define

$$X = \frac{f(C_j)\phi_i - f(C_i)\phi_j}{g(C_j)\phi_i - g(C_i)\phi_j} \tag{3.1}$$

where the $f(C_i)$ and $g(C_i)$ are as yet undetermined functions (to which we will return later) of their respective C_i. Note, however, that since the C_i are not determined better than up to a sign, $f(C_i)$ and $g(C_i)$ must both be even (or possibly both odd, but without loss of generality one can always assume even) functions of their argument.

There exist 28 different ways to write X in terms of the ϕ_i. By equating any two of these expressions we obtain equations for the ϕ_i, i.e. for the product of the τ-functions:

$$\begin{aligned}(f(C_j)g(C_k) - f(C_k)g(C_j))\phi_i + (f(C_k)g(C_i) - f(C_i)g(C_k))\phi_j + (f(C_i)g(C_j)\\ -f(C_j)g(C_i))\phi_k = 0.\end{aligned} \tag{3.2}$$

The overdetermined system of equations (3.2) is a non-autonomous Hirota–Miwa system [8] which describes completely the evolution of the multivariable τ-function in $E_8^{(1)}$. They are, in fact, the bilinear forms of the various equations that 'live' in $E_8^{(1)}$. So far we have not yet examined the question of the consistency of (3.2), which will impose further constraints on the even functions f and g. This will be done in the next section.

For convenience, in what follows and whenever there is no ambiguity, we use the name of a nonlinear variable to mean the point where this variable is defined. Consider the eight NNVs around a given point like $X = (1, 0, 0, 0, 0, 0, 0, 0)$, which, in this particular case happen just to be twice the eight unit vectors of our basis. We can orient seven of them arbitrarily, and then the orientation of the eighth one is fixed, so the sum of the oriented vectors is four times any of the 2^7 (arbitrarily oriented) NVs of half-integer coordinates, along 64 directions. Consider one of these vectors, for instance $\overrightarrow{T} = (1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2)$. We now consider the point $(3/4, -1/4, -1/4, -1/4, -1/4, -1/4, -1/4, -1/4)$ such that the vector from it to the site of X is one-half the NV considered above. It turns out to be a valid nonlinear site where we can define a nonlinear variable Y. This was not *a priori* obvious. For instance, if we translate the site of X by one-half of one of the 56 other NVs, we would not end up at a midpoint of two NNN τ, and no nonlinear variable could be defined there. Similarly to Y we can introduce $\overline{Y}$ corresponding to the point $(5/4, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4)$ such that $\overrightarrow{\overline{Y}X} = -\overrightarrow{T}/2$. Here, the overbar denotes a translation by the full NV, $\overrightarrow{T}$. Since the point $\overline{Y}$ is distant from the site of Y by a full NV, all the τ around $\overline{Y}$ are in the same positions with respect to it as those around Y but not as around X. In fact, one can easily convince oneself that the eight NNVs around Y and $\overline{Y}$ are identical, and have all their coordinates $1/2$, but for one coordinate $-3/2$ at any of the eight positions. They are symmetrical to the NNVs around X with respect to the hyperplane orthogonal to the YX line.

The eight C_i around X, which are the scalar products of the position vector $\overrightarrow{O'X}$ with the appropriate NNVs, are just twice the coordinates, with a shift due to the position of O': $C_i = 2a_i'$, (where $a_i' = a_i - \alpha_i, a_1 = 1, a_j = 0$ for $j \neq 1$). The corresponding quantities $F_j, \overline{F}_j$ around $Y, \overline{Y}$, corresponding to the vectors $\overrightarrow{O'Y}$ and $\overrightarrow{O'\overline{Y}}$, are $F_j = 2\zeta - 2b_j', \overline{F}_j = 2\overline{\zeta} - 2\overline{b}_j'$, with $b_j' = b_j - \alpha_j$ where the b_j are the coordinates of Y, $\zeta = \overrightarrow{O'Y} \cdot \overrightarrow{T}/2 = 1/4 \sum_k b_k'$ and similarly for $\overline{Y}$. In the translation by the full NV, $\overrightarrow{T}$, from Y to $\overline{Y}$ the shift of each b_j is 1/2 and thus the shift of ζ is one ($\overrightarrow{T}$ has squared length 2). So the shift of each F_j is also one. The same shift of one will affect each C_i when translating X by one full $\overrightarrow{T}$. Moreover, if we compute in X the analogue of ζ, namely $z = \overrightarrow{O'X} \cdot \overrightarrow{T}/2 = 1/4 \sum_k a_k'$, we have $\zeta = z - 1/2$, $\overline{\zeta} = z + 1/2$.

Among the 64 distinct NVs around X (or any other point similar to X, for that matter) that allow reaching a nonlinear site like Y, each one is orthogonal to 35 of the others, and has a scalar product ± 1 with the 28 remaining ones. For instance, the NV $\overrightarrow{T} = (1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2)$ is orthogonal to the 35 ones having four coordinates 1/2 and four $-1/2$ (counting opposite vectors only once) and has scalar product 1, say, with the 28 NVs having six coordinates 1/2 and two $-1/2$, defining thus 28 points forming an equilateral triangle with X and Y (and 28 others forming an equilateral triangle with X and $\overline{Y}$). Let us call W a variable defined at one of the sites forming an equilateral triangle with X and Y. To be specific let us choose the point W_{23} such that the vector $\overrightarrow{W_{23}X}$ is one-half the NV with negative signs in second and third positions, $W_{23} = (3/4, 1/4, 1/4, -1/4, -1/4, -1/4, -1/4, -1/4)$. (We chose the second and third positions rather than the first purely for aesthetic reasons, in order to stay as close to the origin as possible, but a W with one index 1 is just as good as any other one, since the origin is by no means a special point). The symmetric $\tilde{W}_{23} = (5/4, -1/4, -1/4, 1/4, 1/4, 1/4, 1/4, 1/4)$ of W_{23} with respect to X is also a valid point to define a nonlinear variable, and forms an equilateral triangle with X and $\overline{Y}$. Note, however, that the points in the XYW_{23} two-dimensional plane that form a regular hexagon of centre X with Y, $\overline{Y}$, W_{23} and $\tilde{W}_{23}$, namely $(1, \epsilon/2, \epsilon/2, 0, 0, 0, 0, 0)$ for $\epsilon = \pm 1$, are *not* midpoints of τ in NNN positions and no nonlinear variables can be defined there.

In order to define a variable like X through (3.1) we need two products ϕ involving four τ. It turns out that just six well chosen τ suffice to define all three variables X, Y and W: the two τ_{+-} and τ_{-+} at $(1/2, 1/2, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2)$ and $(1/2, -1/2, 1/2, -1/2, -1/2, -1/2, -1/2, -1/2)$ (the indices refer to the signs of the second and third coordinates) and the four $\tau_{2,\epsilon}$ and $\tau_{3,\epsilon}$ ($\epsilon = \pm 1$) at the points $(1, \epsilon, 0, 0, 0, 0, 0, 0)$ and $(1, 0, \epsilon, 0, 0, 0, 0, 0)$. Indeed, X is the midpoint of the two NNN pairs $\{\tau_{i+}, \tau_{i-}\}$ $i = 2, 3$ while Y is that of the NNN pairs $\{\tau_{+-}, \tau_{2-}\}$, $\{\tau_{-+}, \tau_{3-}\}$ and W that of the pairs $\{\tau_{+-}, \tau_{3+}\}$ and $\{\tau_{-+}, \tau_{2+}\}$. Note that the vectors $\overrightarrow{\tau_{-+}\tau_{+-}}$, $\overrightarrow{\tau_{2-}\tau_{3-}}$ and $\overrightarrow{\tau_{3+}\tau_{2+}}$ are all equal to the vector $\overrightarrow{S} = (0, 1, -1, 0, 0, 0, 0, 0)$ and that any two of these three vectors form a square. The whole picture is a triangular right prism having the six τ at its vertices. Each basis $\{\tau_{-+}, \tau_{2-}, \tau_{3+}\}$ and $\{\tau_{+-}, \tau_{3-}, \tau_{2+}\}$ of this prism is an equilateral triangle of side $\sqrt{2}$, while the height $\overrightarrow{S}$ has the same length so the three faces are the aforementioned squares having for centres the points X, Y and W_{23} respectively.

Next we compute the C_i corresponding to the pairs around X, scalar products of $\overrightarrow{O'X}$ with the corresponding NNVs $(0, 2, 0, 0, 0, 0, 0, 0)$ and $(0, 0, 2, 0, 0, 0, 0, 0)$ and find $2a_i'$, $i = 2, 3$ respectively. The relevant F_j around Y corresponding to the pairs $\{\tau_{+-}, \tau_{2-}\}$, $\{\tau_{-+}, \tau_{3-}\}$ are $F_2 = 2\zeta - 2b_2'$ and $F_3 = 2\zeta - 2b_3'$. The relevant K_m around W_{23} correspond to the pairs $\{\tau_{-+}, \tau_{2+}\}$, $\{\tau_{+-}, \tau_{3+}\}$ and turn out to be $K_2 = -2z + 1/2 - c_2' + c_3'$ and $K_3 = -2z + 1/2 + c_2' - c_3'$ respectively (again $c_m' = c_m - \alpha_m$ where the c_m are the coordinates of W_{23}). The origin of the

$1/2$ shift comes from the analogue of z computed at W_{23} using the c_m, which turn out to be $z - 1/4$.

Up to this point, this is a purely geometric description. We have not yet expressed the f, g in terms of the C_j. We have X by specifying $i = 3$, $j = 2$ in (3.1)

$$X = \frac{f(C_2)\phi_3 - f(C_3)\phi_2}{g(C_2)\phi_3 - g(C_3)\phi_2} \tag{3.3}$$

with $\phi_i = \tau_{i+}\tau_{i-}$. Solving for the ratio of τ we find

$$\frac{\tau_{2+}\tau_{2-}}{\tau_{3+}\tau_{3-}} = \frac{g(C_2)X - f(C_2)}{g(C_3)X - f(C_3)}. \tag{3.4}$$

Similarly, we have

$$\frac{\tau_{+-}\tau_{2-}}{\tau_{-+}\tau_{3-}} = \frac{g(F_2)Y - f(F_2)}{g(F_3)Y - f(F_3)} \tag{3.5}$$

and

$$\frac{\tau_{-+}\tau_{2+}}{\tau_{+-}\tau_{3+}} = \frac{g(K_2)W_{23} - f(K_2)}{g(K_3)W_{23} - f(K_3)}. \tag{3.6}$$

It is straightforward to eliminate all the τ from (3.4)–(3.6) and find the contiguity relation

$$\frac{g(C_3)X - f(C_3)}{g(C_2)X - f(C_2)}\,\frac{g(F_2)Y - f(F_2)}{g(F_3)Y - f(F_3)}\,\frac{g(K_2)W_{23} - f(K_2)}{g(K_3)W_{23} - f(K_3)} = 1. \tag{3.7}$$

This is what we call a Miura transformation: given any two of the X, Y and W_{23} we can obtain the third one. It is clear from (3.7) that all three variables play a symmetric role. From (3.7) the nonlinear equations satified by Y, $\overline{Y}$ and X can be derived from the analysis of the geometry.

4. Compatibility conditions and the nonlinear equations

We still have not considered the compatibility of the Hirota–Miwa equations (3.2). It will in fact turn out to be simpler to check their consistency on the Miura equations (3.7). Indeed, if two variables are known on two summits of an equilateral triangle of side $\sqrt{2}/2$, the one on the third summit is determined by (3.7). If we consider a tetrahedron of the same side, then any two variables determine both the others, using (3.7) on the two sides which contain the two known variables. But then there are two more sides where all three variables are now determined, and a compatibility condition must be satisfied on them. Such a tetrahedron is, for instance, the one with apices X, Y, W_{23} and W_{24} of coordinates $(3/4, 1/4, -1/4, 1/4, -1/4, -1/4, -1/4, -1/4)$. It turns out that the condition on the even functions f and g for the compatibility to be satisfied is that there exists some odd function h such that

$$f(C)g(D) - f(D)g(C) = h(C + D)h(C - D) \tag{4.1}$$

for all C, D. Obtaining the general solution of (4.1) appears to be a very difficult task. However, we are able to find several interesting solutions. In particular, let us make the simplifying assumption that g is constant (which we can take equal to 1). In this case we can show that (4.1) has only two solutions (up to a rescaling of the dependent and independent variables). The first corresponds to $f(x) \equiv x^2$ and $h(x) \equiv x$, leading to a difference discrete Painlevé equation with seven parameters. The second corresponds to $f(x) \equiv \sinh^2 \lambda x$ and $h(x) \equiv \sinh \lambda x$ and leads to a q-type equation. In these cases (3.7) becomes respectively

$$\frac{X - C_3^2}{X - C_2^2}\,\frac{Y - F_2^2}{Y - F_3^2}\,\frac{W_{23} - K_2^2}{W_{23} - K_3^2} = 1 \tag{4.2}$$

$$\frac{X - \sinh^2 C_3}{X - \sinh^2 C_2}\,\frac{Y - \sinh^2 F_2}{Y - \sinh^2 F_3}\,\frac{W_{23} - \sinh^2 K_2}{W_{23} - \sinh^2 K_3} = 1. \tag{4.3}$$

In the general case we can exhibit one solution, but we cannot prove that it is the only existing one. This solution is expressed in terms of theta functions. Indeed (4.1) is satisfied if we take $f(x) \equiv \theta_1^2(\kappa x|m)$, $g(x) \equiv \theta_0^2(\kappa x|m)$ and $h(x) \equiv \theta_0(0|m)\theta_1(\kappa x|m)$ for arbitrary parameter m. Using these expressions f, g, h one can write the Miura (3.7) in terms of Jacobi elliptic functions only, and the same is true for the nonlinear equation between Y, X and $\overline{Y}$. Indeed, (3.7) becomes (up to a renormalization of X, Y and W_{23})

$$\frac{X-\mathrm{sn}^2C_3}{X-\mathrm{sn}^2C_2}\,\frac{Y-\mathrm{sn}^2F_2}{Y-\mathrm{sn}^2F_3}\,\frac{W_{23}-\mathrm{sn}^2K_2}{W_{23}-\mathrm{sn}^2K_3}=\frac{\theta_0^2(C_2)}{\theta_0^2(C_3)}\,\frac{\theta_0^2(F_3)}{\theta_0^2(F_2)}\,\frac{\theta_0^2(K_3)}{\theta_0^2(K_2)} \tag{4.4}$$

where we have dropped the parameter m. Moreover, one can check that the six quantities C, F, K have zero sum and moreover satisfy the relations $C_2-C_3=F_3-F_2=K_3-K_2$. In this case one can show that the rhs of (4.4) can in fact be written in terms of Jacobi elliptic functions only. We can parametrize the six quantities C, F, K as $C_{3,2}=\alpha-\beta\pm 2s$, $F_{2,3}=\beta-\gamma\pm 2s$, $K_{2,3}=\gamma-\alpha\pm 2s$, with $\alpha+\beta+\gamma=0$, and we find that (4.4) becomes

$$\begin{aligned}\frac{X-\mathrm{sn}^2(\alpha-\beta+2s)}{X-\mathrm{sn}^2(\alpha-\beta-2s)}\,&\frac{Y-\mathrm{sn}^2(\beta-\gamma+2s)}{Y-\mathrm{sn}^2(\beta-\gamma-2s)}\,\frac{W_{23}-\mathrm{sn}^2(\gamma-\alpha+2s)}{W_{23}-\mathrm{sn}^2(\gamma-\alpha-2s)}\\ &=\left(\frac{1-m^2\mathrm{sn}^2(\alpha-s)\mathrm{sn}^2(\beta+s)}{1-m^2\mathrm{sn}^2(\alpha+s)\mathrm{sn}^2(\beta-s)}\,\frac{1-m^2\mathrm{sn}^2(\beta-s)\mathrm{sn}^2(\gamma+s)}{1-m^2\mathrm{sn}^2(\beta+s)\mathrm{sn}^2(\gamma-s)}\right.\\ &\quad\left.\times\frac{1-m^2\mathrm{sn}^2(\gamma-s)\mathrm{sn}^2(\alpha+s)}{1-m^2\mathrm{sn}^2(\gamma+s)\mathrm{sn}^2(\alpha-s)}\right)^2\end{aligned} \tag{4.5}$$

where we can explicitly see that in the limit $m\to 0$ the rhs recovers the value unity.

Suppose we now consider some other equilateral triangle, one summit of which is X, but where Y is not necessarily a summit. Around this triangle we will get an analogue of equation (3.7). In particular, we are interested in the triangle $XW_{23}V$ where V has coordinates $(5/4, 1/4, 1/4, 1/4, -1/4, -1/4, -1/4, -1/4)$ so $\overrightarrow{XV}$ is orthogonal to $\overrightarrow{YX}$. Eliminating W_{23} between the Miura in these two triangles, one can obtain a Miura in the isosceles right triangle YXV. One can easily convince oneself that this relation is still linear separately in Y and V (but no longer in X). On the other hand, the point $\tilde{W}_{78}$ of coordinates $(5/4, 1/4, 1/4, 1/4, 1/4, 1/4, -1/4, -1/4)$ forms an equilateral triangle not only with X and $\overline{Y}$ (as any $\tilde{W}$ does), but also with X and V. So just as in the above construction, one can obtain a Miura in the isosceles right triangle $VX\overline{Y}$, which is linear separately in V and $\overline{Y}$. Eliminating V leads to a relation involving only Y, X and $\overline{Y}$, which is still linear separately in Y and $\overline{Y}$, though not in X. We thus obtain the first half of the nonlinear equation. Working around Y one could have found a similar relation between $\underline{X}$, Y and X, where $\underline{X}$ is the point of coordinates $(1/2, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2, -1/2)$ such that $\overrightarrow{\underline{X}X}$ is one full step $\overrightarrow{T}$.

The construction we just presented allows one to derive the nonlinear equation. It goes without saying that the bulk of computations is considerable and, as a matter of fact, in the case of the elliptic discrete Painlevé equation, prohibitively so. Thus we shall not present its explicit form and limit ourselves to those of the q- and δ-equations. Below we present just their final forms, which are obtained after the appropriate scalings of the dependent and independent variables are introduced.

For the q equation we shall use the notation $q_n=q_0\lambda^n$ and $\rho_n=q_n/\sqrt{\lambda}$. We start from eight constants d_i with the constraint that their product is unity. Let $m_1, m_2, \ldots, m_7$ be the elementary symmetric functions of order 1 to 7, i.e. $m_1=\sum_i d_i$, $m_2=\sum_{i<j} d_i d_j$ (the constraint meaning $m_8=\prod_i d_i=1$) of these eight constants. Then the equations are

$$\frac{(y_{n+1}\rho_{n+1}q_n - x_n)(y_n\rho_n q_n - x_n) - (\rho_{n+1}^2 q_n^2 - 1)(\rho_n^2 q_n^2 - 1)}{(y_{n+1}/(\rho_{n+1}q_n) - x_n)(y_n/(\rho_n q_n) - x_n) - (1 - 1/(\rho_{n+1}^2 q_n^2))(1 - 1/(\rho_n^2 q_n^2))}$$
$$= (x_n^4 - m_1 q_n x_n^3 + (m_2 q_n^2 - 3 - q_n^8)x_n^2 + (m_7 q_n^7 - m_3 q_n^3 + 2m_1 q_n)x_n$$
$$+ q_n^8 - m_6 q_n^6 + m_4 q_n^4 - m_2 q_n^2 + 1)$$
$$\{x_n^4 - m_7 x_n^3/q_n + (m_6/q_n^2 - 3 - 1/q_n^8)x_n^2 + (m_1/q_n^7 - m_5/q_n^3 + 2m_7/q_n)x_n$$
$$+ 1/q_n^8 - m_2/q_n^6 + m_4/q_n^4 - m_6/q_n^2 + 1\}^{-1} \tag{4.6a}$$

$$\frac{(x_{n-1}\rho_n q_{n-1} - y_n)(x_n\rho_n q_n - y_n) - (\rho_n^2 q_{n-1}^2 - 1)(\rho_n^2 q_n^2 - 1)}{(x_{n-1}/(\rho_n q_{n-1}) - y_n)(x_n/(\rho_n q_n) - y_n) - (1 - 1/(\rho_n^2 q_{n-1}^2))(1 - 1/(\rho_n^2 q_n^2))}$$
$$= (y_n^4 - m_7\rho_n y_n^3 + (m_6\rho_n^2 - 3 - \rho_n^8)y_n^2 + (m_1\rho_n^7 - m_5\rho_n^3 + 2m_7\rho_n)y_n$$
$$+ \rho_n^8 - m_2\rho_n^6 + m_4\rho_n^4 - m_6\rho_n^2 + 1)$$
$$\{y_n^4 - m_1 y_n^3/\rho_n + (m_2/\rho_n^2 - 3 - 1/\rho_n^8)y_n^2 + (m_7/\rho_n^7 - m_3/\rho_n^3 + 2m_1/\rho_n)y_n$$
$$+ 1/\rho_n^8 - m_6/\rho_n^6 + m_4/\rho_n^4 - m_2/\rho_n^2 + 1\}^{-1}. \tag{4.6b}$$

For the δ-equation we shall use the notation $z_n = z_0 + n\delta$ and $\zeta_n = z_n - \delta/2$. Here we start from eight constants k_i with the constraint that their sum is zero. Let $s_2, s_3, \ldots, s_8$ be their elementary symmetric functions of order 2 to 8 (from the constraint, $s_1 = \sum_i k_i = 0$). Then the equations are

$$\frac{(x_n - y_{n+1} + (z_n + \zeta_{n+1})^2)(x_n - y_n + (z_n + \zeta_n)^2) + 4x_n(z_n + \zeta_{n+1})(z_n + \zeta_n)}{(z_n + \zeta_n)(x_n - y_{n+1} + (z_n + \zeta_{n+1})^2) + (z_n + \zeta_{n+1})(x_n - y_n + (z_n + \zeta_n)^2)}$$
$$= 2\frac{x_n^4 + S_2 x_n^3 + S_4 x_n^2 + S_6 x_n + S_8}{8z_n x_n^3 + S_3 x_n^2 + S_5 x_n + S_7} \tag{4.7a}$$

where the S_i are the elementary symmetric functions of the quantities $k_i + z_n$ (which are essentially what was called C_i in section 3), so $S_2 = 28z_n^2 + s_2$, $S_3 = 56z_n^3 + 6z_n s_2 + s_3$, etc (and $8z_n = S_1$);

$$\frac{(y_n - x_{n-1} + (z_{n-1} + \zeta_n)^2)(y_n - x_n + (z_n + \zeta_n)^2) + 4y_n(z_n + \zeta_n)(z_{n-1} + \zeta_n)}{(z_n + \zeta_n)(y_n - x_{n-1} + (z_{n-1} + \zeta_n)^2) + (z_{n-1} + \zeta_n)(y_n - x_n + (z_n + \zeta_n)^2)}$$
$$= 2\frac{y_n^4 + \Sigma_2 y_n^3 + \Sigma_4 y_n^2 + \Sigma_6 y_n + \Sigma_8}{8\zeta_n y_n^3 + \Sigma_3 y_n^2 + \Sigma_5 y_n + \Sigma_7} \tag{4.7b}$$

where the Σ_i are the elementary symmetric functions of the quantities $\zeta_n - k_i$ (which are essentially the F_i of section 3), so $\Sigma_2 = 28\zeta_n^2 + s_2$, $\Sigma_3 = 56\zeta_n^3 + 6\zeta_n s_2 - s_3$, etc (and $8\zeta_n = \Sigma_1$).

System (4.7) can be obtained from (4.6) by a coalescence process. Here we shall follow the convention [2] of using upper-case letters for the 'higher' equation, here (4.6), and lower-case letters for 'lower', here (4.7). Indeed, we take $Q_0 = \mathrm{e}^{\epsilon z_0}$, $\Lambda = 1 + \epsilon\delta$, $X = 2 + \epsilon^2 x$, $Y = 2 + \epsilon^2 y$, $D_i = \mathrm{e}^{\epsilon k_i}$. In the limit $\epsilon \to 0$ (so that from q and ρ we obtain $(Q_n - 1)/\epsilon \to z_n$, $(R_n - 1)/\epsilon \to \zeta_n$) we recover (4.7) for x and y. This calculation is quite delicate since the first few orders in the expansions of numerators and denominators on both sides of (4.6) vanish and one has to go up to order 8 in ϵ before finding all significant quantities.

Another coalescence can lead from (4.6) to a known q-$\mathrm{P_{VI}}$ equation related to the affine Weyl group $E_7^{(1)}$. We take $X = \Omega x$, $Y = \Omega y$, with $\Omega \to \infty$. Among the eight quantities D_i we take four large ones ($\propto\Omega$), and four small ones ($\propto 1/\Omega$). Then the elementary symmetric functions behave, at the dominant term, like powers of Ω. In fact, up to such powers, M_1, M_2, M_3 become the three first elementary symmetric functions m_1, m_2, m_3 of the four 'large' D_i, and M_7, M_6 M_5 those, namely n_1, n_2, n_3, of the *inverse* of the four 'small' ones, while M_4 becomes the common value p of the products. At the limit, keeping only the dominant terms, (4.6) becomes

$$\frac{(y_{n+1}R_{n+1}Q_n - x_n)(y_nR_nQ_n - x_n)}{(y_{n+1}/(R_{n+1}Q_n) - x_n)(y_n/(R_nQ_n) - x_n)} = \frac{x_n^4 - m_1Q_nx_n^3 + m_2Q_n^2x_n^2 - m_3Q_n^3x_n + pQ_n^4}{x_n^4 - n_1x_n^3/Q_n + n_2x_n^2/Q_n^2 - n_3x_n/Q_n^3 + p/Q_n^4} \tag{4.8a}$$

$$\frac{(x_{n-1}R_nQ_{n-1} - y_n)(x_nR_nQ_n - y_n)}{(x_{n-1}/(R_nQ_{n-1}) - y_n)(x_n/(R_nQ_n) - y_n)} = \frac{y_n^4 - n_1R_ny_n^3 + n_2R_n^2y_n^2 - n_3R_n^3y_n + pR_n^4}{y_n^4 - m_1y_n^3/R_n + m_2y_n^2/R_n^2 - m_3y_n/R_n^3 + p/R_n^4}. \tag{4.8b}$$

Then let us replace the y by their inverse. System (4.8) becomes

$$\frac{(R_{n+1}Q_n - x_ny_{n+1})(R_nQ_n - x_ny_n)}{(1/(R_{n+1}Q_n) - x_ny_{n+1})(1/(R_nQ_n) - x_ny_n)} = \frac{x_n^4 - m_1Q_nx_n^3 + m_2Q_n^2x_n^2 - m_3Q_n^3x_n + pQ_n^4}{x_n^4 - n_1x_n^3/Q_n + n_2x_n^2/Q_n^2 - n_3x_n/Q_n^3 + p/Q_n^4} \tag{4.9a}$$

$$\frac{(x_{n-1}y_nR_nQ_{n-1} - 1)(x_ny_nR_nQ_n - 1)}{(x_{n-1}y_n/(R_nQ_{n-1}) - 1)(x_ny_n/(R_nQ_n) - 1)} = \frac{1 - n_1R_ny_n + n_2R_n^2y_n^2 - n_3R_n^3y_n^3 + y_n^4pR_n^4}{1 - m_1y_n/R_n + m_2y_n^2/R_n^2 - m_3y_n^3/R_n^3 + y_n^4p/R_n^4}. \tag{4.9b}$$

Inverting both sides of (4.9b), gauging the x and y through $x_n \to x_np^{1/4}/Q_n$, $y_n \to y_np^{-1/4}/R_n$ and redefining $q_n = Q_n^2$, $\rho_n = R_n^2$ we obtain the system

$$\frac{(x_ny_{n+1} - q_n\rho_{n+1})(x_ny_n - q_n\rho_n)}{(x_ny_{n+1} - 1)(x_ny_n - 1)} = \frac{x_n^4 - m_1q_nx_n^3 + m_2q_n^2x_n^2 - m_3q_n^3x_n + q_n^4}{x_n^4 - n_1x_n^3 + n_2x_n^2 - n_3x_n + 1} \tag{4.10a}$$

$$\frac{(x_{n-1}y_n - q_{n-1}\rho_n)(x_ny_n - q_n\rho_n)}{(x_{n-1}y_n - 1)(x_ny_n - 1)} = \frac{y_n^4 - m_3\rho_ny_n^3 + m_2\rho_n^2y_n^2 - m_1\rho_n^3y_n + \rho_n^4}{y_n^4 - n_3y_n^3 + n_2y_n^2 - n_1y_n + 1} \tag{4.10b}$$

which is the equation we introduced in [11] under the name of asymmetric q-$\mathrm{P_{VI}}$. From (4.7) a similar coalescence would lead to the other equation associated with the affine Weyl group $E_7^{(1)}$ and introduced in [11], namely the asymmetric d-$\mathrm{P_V}$.

Before completing this section we mention one last degeneration, that of the elliptic equation towards the q equation. Since we have not given the explicit form of the elliptic-discrete $\mathbb{P}$ we shall present the coalescence at the level of the Miura transformations. We start from (4.4) and consider the limit $m \to 0$. At this limit the elliptic sines go over to circular sines and moreover $\theta_0 \to 1$. Thus, taking $\kappa = \mathrm{i}\lambda$ (and with a sign change of X, Y, W_{23}) we recover exactly (4.3).

While all the discrete Painlevé equations obtained here have eight parameters, their continuous limit is just $\mathrm{P_{VI}}$ (which has four parameters and one continuous independent variable). As a matter of fact, all discrete $\mathbb{P}$ associated with the affine Weyl groups $E_8^{(1)}$, $E_7^{(1)}$ and $E_6^{(1)}$ [14] have $\mathrm{P_{VI}}$ as the continuous limit (although they contain more parameters than $\mathrm{P_{VI}}$, to begin with). On the other hand, the asymmetric q-$\mathrm{P_{III}}$ equation [15], described by the group D_5, contains exactly the same number of parameters as $\mathrm{P_{VI}}$ and was, in fact, historically the first discrete form of $\mathrm{P_{VI}}$ discovered.

5. Conclusion

In this paper we have presented the geometric construction of the eight-parameter discrete Painlevé equation. This approach, based on affine Weyl groups, is particularly interesting in the present case because, given the complexity of the equations, there is no possibility to obtain them through a brute-force calculation. As a matter of fact, this is the very first instance where the geometrical approach has allowed construction of a previously unknown discrete Painlevé equation.

One important result obtained here, and which is unique (in the sense that it cannot exist for d-$\mathbb{P}$ not described in $E_8^{(1)}$), is the construction of elliptic-discrete $\mathbb{P}$. Their existence was first proven rigorously by Sakai in [7]. Here we have presented the explicit construction in the bilinear case and also up to the Miura level for the nonlinear variables. However, the complexity (and sheer bulk) of computations did not allow us to produce the explicit form of the elliptic d-$\mathbb{P}$ in nonlinear variables.

Having obtained the basic discrete Painlevé equations does not exhaust the possibilities related to the geometry of $E_8^{(1)}$. It is possible, within the same space of the weights of $E_8^{(1)}$, to define evolutions along more complicated paths and obtain more second-order discrete $\mathbb{P}$ (just as we have done for simpler Weyl groups). Given the richness of the $E_8^{(1)}$ group this is a project that must be undertaken with extreme care. We intend to return to this question in some future work, once the analogous studies in $E_7^{(1)}$ and $E_6^{(1)}$ have first been carried through.

Acknowledgment

The authors are grateful to Dr H Sakai for illuminating discussions.

References

[1] Ramani A, Grammaticos B and Hietarinta J 1991 *Phys. Rev. Lett.* **67** 1829
[2] Ramani A and Grammaticos B 1996 *Physica* A **228** 160
[3] Okamoto K 1981 *Physica* D **2** 525
[4] Ramani A, Ohta Y, Satsuma J and Grammaticos B 1998 *Commun. Math. Phys.* **192** 67
[5] Ramani A and Grammaticos B 1996 The Grand Scheme for discrete Painlevé equations *Lecture at the Toda Symp.*
[6] Grammaticos B and Ramani A 2000 *Reg. Chaot. Dyn.* **5** 53
[7] Sakai H 1999 Rational surfaces associated with affine root systems and geometry of the Painlevé equations *PhD Thesis* Kyoto University
[8] Hirota R 1981 *J. Phys. Soc. Japan* **50** 3785
Miwa T 1982 *Proc. Japan. Acad.* **58** 9
[9] Grammaticos B, Ramani A and Papageorgiou V 1991 *Phys. Rev. Lett.* **67** 1825
[10] Ohta Y, Tamizhmani K M, Grammaticos B and Ramani A 1999 *Phys. Lett.* A **262** 152
[11] Grammaticos B and Ramani A 1999 *Phys. Lett.* A **257** 288
[12] Ramani A, Grammaticos B and Tamizhmani T 2000 *J. Phys. A: Math. Gen.* **33** 3033
[13] Ramani A, Grammaticos B and Ohta Y 2001 *Commun. Math. Phys.* **217** 315
[14] Grammaticos B, Ramani A and Ohta Y 2001 A unified description of the asymmetric q-P_V and d-P_{IV} equations and their Schlesinger transformations, at press
[15] Jimbo M and Sakai H 1996 *Lett. Math. Phys.* **38** 145